Hamilton-Jacobi Modelling of Relative Motion for Formation Flying

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Abstract

A precise analytical model for the relative motion of a group of satellites in slightly elliptic orbits is introduced. To this aim, we describe the relative motion of an object relative to a circular or slightly elliptic reference orbit in the rotating Hill frame via a low-order Hamiltonian, and solve the Hamilton-Jacobi equation. This results in a first-order solution to the relative motion identical to the Clohessy-Wiltshire approach; here, however, rather than using initial conditions as our constants of the motion, we utilize the canonical momenta and coordinates. This allows us to treat perturbations in an identical manner as in the classical Delaunay formulation of the two-body problem. A precise analytical model for the base orbit is chosen with the included effect of zonal harmonics (J_2, J_3, J_4). A Hamiltonian describing the real relative motion is formed and by differing this from the nominal Hamiltonian, the perturbing Hamiltonian is obtained. Using Hamilton's equations, the variational equations for the new constants are found.

In a manner analogous to the center manifold reduction procedure, the non-periodic part of the motion is canceled through a magnitude analysis leading to simple boundedness conditions that cancel the drift terms due to the higher order perturbations. Using this condition, the variational equations are integrated to give periodic solutions which closely approximate the results from numerical integration (1mm/per orbit for higher order and eccentricity perturbations and 30cm/per orbit for zonal perturbations). This procedure provides a compact and insightful analytical description of the resulting relative motion.

Key words: Formation flying; Canonical transformations; Hamiltonian dynamics.

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1 Introduction

The analysis of relative spacecraft motion constitutes an issue of increasing interest due to exiting and planned spacecraft formation flying and orbital rendezvous missions. It was in the early 60's that Clohessy and Wiltshire first published their celebrated work that utilized a Hill-like rotating Cartesian coordinate system to derive expressions for the relative motion between satellites in the context of a rendezvous problem [1]. The Clohessy-Wiltshire (CW) linear formulation assumed small deviations from a circular reference orbit and used the initial conditions as the constants of the unperturbed motion. Since then, recognizing the limitations of this approach, others have generalized the CW equations for eccentric reference orbits [16,17], and to include perturbed dynamics [8,10,22].

An important modification of the CW linear solution is the use of orbital elements as constants of motion instead of the Cartesian initial conditions. This concept, originally suggested by Hill [2], has been widely used both in the analysis of relative spacecraft motion[3] and in dynamical astronomy [4]. Using this approach allows the effects of orbital perturbations on the relative motion to be examined via variational equations such as Lagrange's planetary equations (LPEs) or Gauss's variational equations (GVEs). Moreover, utilizing orbital elements facilitates the derivation of high-order, nonlinear extensions to the CW solution [5].

There have been a few reported efforts to obtain high-order solutions to the relative motion problem. Recently, Karlgaard and Lutze [7] proposed formulating the relative motion in spherical coordinates in order to derive second-order expressions. The use of Delaunay elements has also been proposed. For instance, Alfriend et al. [8] derived differential equations in order to incorporate perturbations and high-order nonlinear effects into the modelling of relative dynamics.

The CW equations, obtained by utilizing Cartesian coordinates to model the relative motion state-space dynamics, usually cannot be solved in closed-form for arbitrary generalized perturbing forces; on the other hand, the orbital elements or Delaunay-based representations can be straightforwardly expanded to treat orbital perturbations, but they utilize characteristics of the inertial, absolute orbits. Hence, using orbital elements or Delaunay variables constitutes an indirect representation of the relative motion problem.

In [6] we describe a new approach to treating relative motion that brings together the merits of the CW and the orbital elements-based approaches. We develop a Hamiltonian methodology that models the relative motion dynamics using canonical coordinates. This procedure, via the solution of the Hamilton-Jacobi equation, is identical to that leading to the classical Delaunay variables, except that it is performed to first order in the rotating Hill frame. The Hamiltonian formulation facilitates the modelling of high-order terms and orbital perturbations via the variation of parameters while allowing us to obtain closed-form solutions for the relative motion.

We start by deriving the Lagrangian for the relative motion in Cartesian coordinates. Then, using a Legendre transformation, we calculate the Hamiltonian for the relative motion. We partition the Hamiltonian into a linear term and a high-order term. We then solve the Hamilton-Jacobi (HJ) equations for the

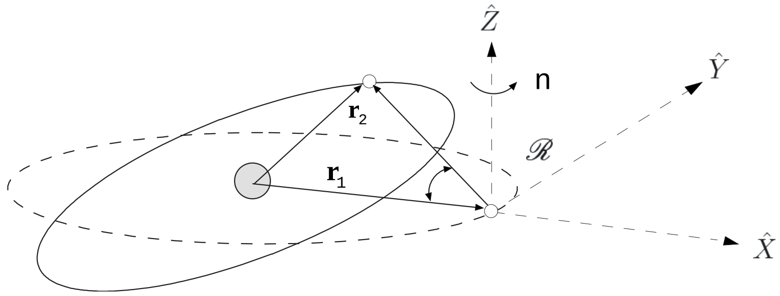

Fig. 1. Relative motion rotating Euler-Hill reference frame

linear part by separation, obtaining new constants for the relative motion which we call epicyclic elements. These elements can then be used to define the parameters of a relative motion orbit and, more importantly, to predict the effects of perturbations via variation of parameters.

In the next section, we briefly summarize the development of the H-J solution and variational equations from [6]. We then follow with three examples of gravitational perturbation. In Section 3, we summarize our results from [6] for the J_2 perturbation on a satellite formation. In Section 4, we use this technique to solve for the effect of the earth oblateness perturbation to the second order (J_2 and ${J_2}^2$) in the potential expansion. We then derive the boundedness condition and analyze the possible periodic orbits. Finally, in section 5, we extend these results to include the effects of up to the fourth order zonal harmonics (J_2, J_3 and J_4).

2 The Hamilton-Jacobi Solution of Relative Motion

For this study, we are considering the motion of a satellite in a Cartesian Euler-Hill frame relative to a circular orbit as shown in Figure 1. Traditionally, relative motion in this frame has been modeled using the Clohessy-Wiltshire (CW) equations via a first-order, linear analysis:

$$\ddot{x} - 2n\dot{y} - 3n^2 x = Q_x \tag{1}$$
$$\ddot{y} + 2n\dot{x} = Q_y \tag{2}$$
$$\ddot{z} + n^2 z = Q_z \tag{3}$$

where (Q_x, Q_y, Q_z) represent small perturbing forces and n is the reference orbit rate.

In the absence of perturbing forces, it is well known that the solution to these equations consists of an elliptical trajectory about the origin with a possible long term drift. The drift can be eliminated by the no-drift constraint, $\dot{y} + 2nx = 0$.

In this work, we approach the problem slightly differently by first formulating the Lagrangian of the motion in the rotating frame (where we have expanded the potential in terms of Legendre polynomials),

$$\mathcal{L} = \frac{1}{2}\left\{(\dot{x} - ny)^2 + (\dot{y} + nx + na)^2 + \dot{z}^2\right\}$$
$$+ n^2 a^2 \sum_{k=0}^{\infty} P_k(\cos\alpha)\left(\frac{\rho}{a}\right)^k - U_p \quad (4)$$

and U_p is a perturbing potential. Our goal is to formulate a Hamiltonian for the entire system, $\mathcal{H} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^1$, that we can partition into a nominal Hamiltonian, $\mathcal{H}^{(0)}$, with which we can solve the Hamilton-Jacobi (H-J) equation, and a perturbing Hamiltonian, $\mathcal{H}^{(1)}$,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \quad (5)$$

By solving the H-J equation on $\mathcal{H}^{(0)}$, we find a set of canonical momenta and coordinates for which $\mathcal{H}^{(0)}$ is a constant. Thus, perturbations can be treated as causing first-order variations of these new coordinates via Hamilton's equations on the perturbing Hamiltonian. This same procedure is followed in the two-body problem to derive Delaunay variables and the corresponding variational equations.

The first step is to drop the perturbing potentials (which include the higher-order terms in the nominal potential), just as in the treatment leading to the Clohessy-Wiltshire equations for relative motion. This is equivalent to only examining small deviations from the reference orbit. We do this by expanding the potential term to second-order to find the low order Lagrangian,

$$\bar{\mathcal{L}}^{(0)} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + ((x+1)\dot{y} - y\dot{x}) + \frac{3}{2} + \frac{3}{2}x^2 - \frac{1}{2}z^2 \quad (6)$$

Where we have also normalized rates by n so that time is in units of radians, or the argument of latitude, and we have normalized distances by the reference orbit semi-major axis, a. Not surprisingly, applying the Euler-Lagrange equations to this Lagrangian results in the expected C-W equations, Eqs. (1) - (3).

For brevity, we do not repeat the entire H-J solution procedure here. Details can be found in [6]. Using the Lagrangian in Eq. (6) we find the canonical momenta,

$$p_x = \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{x}} = \dot{x} - y$$
$$p_y = \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{y}} = \dot{y} + x + 1 \quad (7)$$
$$p_z = \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{z}} = \dot{z}$$

and the corresponding unperturbed Hamiltonian,

$$\mathcal{H}^{(0)} = \frac{1}{2}(p_x + y)^2 + \frac{1}{2}(p_y - x - 1)^2 + \frac{1}{2}p_z^2 - \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 \tag{8}$$

This Hamiltonian is used to solve the H-J equation, resulting in a new set of canonical momenta, $(\alpha_1, \alpha_2, \alpha_3)$, and canonical coordinates, (Q_1, Q_2, Q_3),

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(p_x + y)^2 + 2(p_y - x - 1)^2 + \frac{9}{2}x^2 + 6x(p_y - x - 1) \\
&= \frac{1}{2}\left(\dot{x}^2 + (2\dot{y} + 3x)^2\right) && (9) \\
\alpha_2 &= \frac{1}{2}p_z^2 + \frac{1}{2}z^2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2 && (10) \\
\alpha_3 &= p_y + x - 1 = \dot{y} + 2x && (11) \\
Q_1 &= u - u_0 + \beta_1 = \tan^{-1}\left(\frac{x - 2\alpha_3}{\sqrt{2\alpha_1 - 4\alpha_3^2 + 4\alpha_3 x - x^2}}\right) \\
&= -\tan^{-1}\left(\frac{3x + 2\dot{y}}{\dot{x}}\right) && (12) \\
Q_2 &= u - u_0 + \beta_2 = \tan^{-1}\left(\frac{z}{\sqrt{2\alpha_2 - z^2}}\right) \\
&= \tan^{-1}\left(\frac{z}{\dot{z}}\right) && (13) \\
Q_3 &= -3\alpha_3(u - u_0) + \beta_3 = y - 2\sqrt{2\alpha_1 - 4\alpha_3^2 + 4\alpha_3 x - x^2} \\
&= -2\dot{x} + y && (14)
\end{aligned}$$

where $(\beta_1, \beta_2, \beta_3)$ are constants and the Hamiltonian in the new coordinates is,

$$\mathcal{H}^{(0)} = \alpha_1 + \alpha_2 \tag{15}$$

In the absence of perturbations, these new coordinates are, of course, constant and given by the Cartesian initial conditions via Eqs. (9) - (14). We call them "epicyclic elements" as they describe epicycle-like motion about a reference circular orbit. These equations can then be solved for x, y, and z to yield the cartesian generating solution in the Hill frame,

$$\begin{aligned}
x(t) &= 2\alpha_3 + \sqrt{2\alpha_1}\sin(Q_1) = 2\alpha_3 + \sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) && (16) \\
y(t) &= Q_3 + 2\sqrt{2\alpha_1}\cos(Q_1) = -3\alpha_3(u - u_0) + \beta_3 + 2\sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) && (17) \\
z(t) &= \sqrt{2\alpha_2}\sin(Q_2) = \sqrt{2\alpha_2}\sin(u - u_0 + \beta_2) && (18)
\end{aligned}$$

It is also straightforward to find expressions for the cartesian rates and the canonical momenta in terms of these new variables. Eqs. (16) - (17) are the same elliptic motion solution as one gets from the C-W equations, except here written in terms of the new elements rather than Cartesian initial conditions. The value of this approach is in the canonicity of these elements. Because they solve the H-J equation, if we write the perturbing Hamiltonian in terms of them, we find that their variation under perturbations is given by the first-order Hamilton's equations,

$$\dot{\alpha}_i = -\frac{\partial \mathcal{H}^{(1)}}{\partial Q_i} \tag{19}$$

$$\dot{\beta}_i = \frac{\partial \mathcal{H}^{(1)}}{\partial \alpha_i} \tag{20}$$

$$\dot{Q}_i = \frac{\partial \mathcal{H}^{(0)}}{\partial \alpha_i} + \dot{\beta}_i \tag{21}$$

Before proceeding with our treatment of perturbations, there is one more helpful simplification. The epicyclic elements above parameterize the motion in terms of amplitude and phase. Also, as the α's enter in as square roots, the variational equations can become quite complicated (and often singular). It is therefore often more convenient to introduce an alternative, amplitude like set via the canonical transformation,

$$a_1 = \sqrt{2\alpha_1}\cos\beta_1 \tag{22}$$

$$b_1 = \sqrt{2\alpha_1}\sin\beta_1 \tag{23}$$

$$a_2 = \sqrt{2\alpha_2}\cos\beta_2 \tag{24}$$

$$b_2 = \sqrt{2\alpha_2}\sin\beta_2 \tag{25}$$

$$a_3 = \alpha_3 \tag{26}$$

$$b_3 = \beta_3 \tag{27}$$

It can be shown that this set arises from two symplectic transformations from (α_i, Q_i). Thus, the variations of these new variables are also given by Hamilton's equations on the perturbing Hamiltonian. We call these new elements *contact epicyclic elements*. The new Cartesian position equations in terms of the contact elements become,

$$x(t) = 2a_3 + a_1\sin(u - u_0) + b_1\cos(u - u_0) \tag{28}$$

$$y(t) = b_3 - a_3(u - u_0) - 2b_1\sin(u - u_0) + 2a_1\cos(u - u_0) \tag{29}$$

$$z(t) = b_2\cos(u - u_0) + a_2\sin(u - u_0) \tag{30}$$

3 The First-Order Oblateness Perturbation

In Ref. [6] we go into some detail on using this approach to studying the perturbation of a relative motion trajectory due to the J_2 oblateness. We therefore only summarize the results here. For illustration, it is simpler to begin with the restricted case of a circular, equatorial reference orbit and follow the perturbation analysis as above. The perturbing Hamiltonian is,

$$\mathcal{H}^{(1)} = \frac{n^2 J_2 R_\oplus^2 (2z^2 - 1 - 2x - x^2 - y^2)}{2a^2(1 + 2x + x^2 + y^2 + z^2)^{(5/2)}} \tag{31}$$

where we have again normalized distances by the reference orbit radius, a. This is expanded to second-order and (x, y, z) is replaced by their time varying solution in terms of the epicyclic elements. Hamilton's equations are then used to find the variational equations for the elements, which, in terms of the contact elements, are,

$$\dot{a}_1 = \frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \begin{pmatrix} -\cos(u - u_0) + 4\sin(2(u - u_0))a_1 \\ +4\cos(2(u - u_0))b_1 + 2\sin(u - u_0)q_3 \\ +8\cos(u - u_0)a_3 \end{pmatrix} \tag{32}$$

$$\dot{b}_1 = \frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \begin{pmatrix} \sin(u - u_0) + 4\cos(2(u - u_0))a_1 \\ -4\sin(2(u - u_0))b_1 + 2\cos(u - u_0)q_3 \\ -8\sin(u - u_0)a_3 \end{pmatrix} \tag{33}$$

$$\dot{a}_2 = -\frac{9}{4} J_2 \left(\frac{R_\oplus}{a}\right)^2 ((1 + \cos(2(u - u_0)))b_2 - \sin(2(u - u_0))a_2) \tag{34}$$

$$\dot{b}_2 = \frac{9}{4} J_2 \left(\frac{R_\oplus}{a}\right)^2 (\sin(2(u - u_0))b_2 - (1 - \cos(2(u - u_0)))a_2) \tag{35}$$

$$\dot{a}_3 = -\frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 (q_3 + 2\cos(u - u_0)a_1 - 2\sin(u - u_0)b_1) \tag{36}$$

$$\dot{q}_3 = -3a_3 + 3J_2 \left(\frac{R_\oplus}{a}\right)^2 \begin{pmatrix} 1 - 8a_3 - 4\sin(u - u_0)a_1 \\ -4\cos(u - u_0)b_1 \end{pmatrix} \tag{37}$$

Since we are performing our analysis to first-order in J_2 only and we assume small relative motion, these can be simplified in a low-order analysis to,

$$\dot{a}_1 = -\frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \cos(u - u_0) \tag{38}$$

$$\dot{b}_1 = \frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \sin(u - u_0) \tag{39}$$

$$\dot{a}_2 = 0 \tag{40}$$

$$\dot{b}_2 = 0 \tag{41}$$

$$\dot{a}_3 = 0 \tag{42}$$

$$\dot{q}_3 = -3a_3 + 3J_2 \left(\frac{R_\oplus}{a}\right)^2 \tag{43}$$

These equations can be easily solved by quadrature,

$$a_1 = a_1(0) - \frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \sin(u - u_0) \tag{44}$$

$$b_1 = b_1(0) - \frac{3}{2} J_2 \left(\frac{R_\oplus}{a}\right)^2 \cos(u - u_0) \tag{45}$$

$$a_2 = a_2(0) \tag{46}$$

$$b_2 = b_2(0) \tag{47}$$

$$a_3 = a_3(0) \tag{48}$$

$$q_3 = q_3(0) + 3(J_2 \left(\frac{R_\oplus}{a}\right)^2 - a_3(0))(u - u_0) \tag{49}$$

In order to eliminate the along-track drift, we set $a_3(0) = J_2 \left(\frac{R_\oplus}{a}\right)^2$. If we also simplify by considering only in-plane motion (by setting $a_2(0) = b_2(0) = 0$), then inserting these solutions back into Eqs. (28) - (30) shows that one equilibrium solution consists of a constant radial offset of $x = \frac{J_2}{2} \left(\frac{R_\oplus}{a}\right)^2$. This is the same as the well known result for the needed constant radial offset to establish a circular, equatorial orbit in the presence of J_2 (most simply found by equating the gravitational and centrigual forces). This is a convincing validation of the approach.

The true power of the technique is displayed for the general J_2 perturbation problem. Here we find a very simple periodic relative motion condition for any reference orbit at all inclinations. However, to do so we must introduce one complication. We know from the perturbed two-body problem that any satellite orbit will have a long term, secular drift in the node angle and argument of perigee induced by oblateness. Thus, it is clearly impossible, using any technique, to find a boundedness condition for motion relative to a fixed reference orbit (and, of course, the drift will quickly invalidate the small motion assumption). One approach is to treat the perturbation inertially. Schaub and Alfriend [10,8], for example, realizing this, derived general J_2–invariant (and almost invariant) satellite formations by matching the drifts among the satellites. In other words, the satellite orbits still drift relative to the usual

Hill reference frame, but they drift in such a way that the formation remains bounded. Unfortunately, this loses the advantage provided by the relative frame description and tends to have singularity problems.

To solve the problem in our canonical formalism, we return to the original solution of the H-J equation and replace the fixed, Hill-like reference orbit with a circular orbit that also rotates at the mean J_2 induced drift rate. Thus, the new reference frame, rather than rotating only about the z-axis at the nominal orbit rate, now has the more complicated angular velocity,

$$
{}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{R}} = \begin{bmatrix} \dot{\Omega}\sin i \sin u \\ \dot{\Omega}\sin i \cos u \\ \dot{\Omega}\cos i + \dot{u} \end{bmatrix}_{\mathscr{I}} \tag{50}
$$

where u is the argument of latitude, $\dot{u} = n + \delta n$ is the modified orbit rate including the J_2 perturbation, and $n = \sqrt{\mu/\bar{a}^3}$, $\bar{a}$ being the mean semi-major axis. The equations for the drift rates are somewhat subtle, as the usual expressions are written in terms of the initial or mean semi-major axis of the osculating orbit (see, e.g., [11] or [13]). Here, however, we select a circular reference orbit with the radius, $\bar{r}$, equivalent to the mean radius of the J_2 perturbed orbit [12],

$$
\bar{r} = \bar{a} + \frac{3J_2{R_\oplus}^2}{4\bar{a}}(3\sin^2 i - 2) \tag{51}
$$

Since we are free to select the reference orbit, this equation is solved for $\bar{a}$ and then used to find the mean rates of change of the node angle and argument of latitude [12,13] for the arbitrary, circular reference orbit,

$$
\dot{\Omega} = -\frac{3}{2}\bar{n}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \cos i \tag{52}
$$

$$
\delta n = \frac{3}{4}\bar{n}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \left(3 - \frac{7}{2}\sin^2 i\right) \tag{53}
$$

where $\bar{n} = \sqrt{\mu/\bar{r}^3}$. Note also that in Eq. (53) we have included in $\dot{u}$ both the effect of the rate of change of true anomaly and of the argument of perigee as the reference orbit is circular (i.e., $\dot{u} = \dot{M} + \dot{\omega}$).

This angular velocity is then used to find the inertial velocity of the satellite and then kinetic and potential energies. This results in the new, normalized Lagrangian,

$$
\begin{aligned}
\bar{\mathcal{L}} = {} & \frac{1}{2}|\mathbf{v}^{(0)}|^2 + \mathbf{v}^{(0)} \cdot \mathbf{v}^{(1)} + \frac{1}{2}|\mathbf{v}^{(1)}|^2 \\
& + \sum_{k=0}^{\infty} P_k(\cos\alpha)\rho^k - \bar{U}_{zonal}
\end{aligned} \tag{54}
$$

where $\mathbf{v}^{(0)}$ is the part of the normalized velocity in the relative motion frame independent of J_2 (and the same as the velocity in the original problem),

$$\mathbf{v}^{(0)} = \begin{bmatrix} \dot{x} - y \\ \dot{y} + (x+1) \\ \dot{z} \end{bmatrix} \tag{55}$$

and $\mathbf{v}^{(1)}$ is the small remaining term of order J_2,

$$\mathbf{v}^{(1)} = \begin{bmatrix} v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \end{bmatrix} = \begin{bmatrix} \bar{\dot{\Omega}} s_i c_u z - (\bar{\dot{\Omega}} c_i - \bar{\delta n}) y \\ (\bar{\dot{\Omega}} c_i - \bar{\delta n})(x+1) - \bar{\dot{\Omega}} s_i s_u z \\ \bar{\dot{\Omega}} s_i s_u y - \bar{\dot{\Omega}} s_i c_u (x+1) \end{bmatrix} \tag{56}$$

where $\bar{\dot{\Omega}} = \dot{\Omega}/\bar{n}$ and $\bar{\delta n} = \delta n/\bar{n}$.

As we did before, we can expand this Lagrangian and keep only the low order terms (including terms to first-order only in J_2). This allows us to drop the second-order term, $\frac{1}{2}|\mathbf{v}^{(1)}|^2$, and rewrite the Lagrangian,

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}^{(0)} + \mathbf{v}^{(0)} \cdot \mathbf{v}^{(1)} - \bar{U}_{zonal} \tag{57}$$

where we are ignoring the second-order terms of the previous section. It is interesting to note that this Lagrangian could be used in the Euler-Lagrange equations to find second-order equations of motion in this new rotating and drifting frame, which may have some usefulness for control design.

With this Lagrangian, we compute the new canonical momenta,

$$\begin{aligned} p_x &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{x}} = \dot{x} - y + v_x^{(1)} \\ p_y &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{y}} = \dot{y} + x + 1 + v_y^{(1)} \\ p_z &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{z}} = \dot{z} + v_z^{(1)} \end{aligned} \tag{58}$$

and, again using the Legendre transformation, $\mathcal{H} = \sum \dot{q}_i p_i - L$, we find the new Hamiltonian,

$$\begin{aligned} \mathcal{H} = {} & \frac{1}{2}(p_x + y - v_x^{(1)})^2 + \frac{1}{2}(p_y - (x+1) - v_y^{(1)})^2 + \frac{1}{2}(p_z - v_z)^2 \\ & - \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 + y v_x^{(1)} - (x+1) v_y^{(1)} + \bar{U}_{zonal} \end{aligned} \tag{59}$$

Multiplying out the terms in Eq. (59) results in the same low order Hamiltonian, $\mathcal{H}^{(0)}$, as the original problem and the perturbing Hamiltonian,

$$\mathcal{H}^{(1)} = -p_x v_x^{(1)} - p_y v_y^{(1)} - p_z v_z^{(1)} + \bar{U}_{zonal} \tag{60}$$

where we have again dropped terms of second-order (or higher) in J_2. The solution to the H-J equation is the same as before, with the Cartesian relative motion given by Eqs. (28) - (30) in terms of the contact epicyclic elements, only now the motion is referred to the rotating and drifting reference orbit. However, due to the modified definition of the canonical momenta, the cartesian rates are slightly different,

$$\dot{x}(u) = a_1 \cos(u - u_0) - b_1 \sin(u - u_0) - v_x^{(1)} \tag{61}$$

$$\dot{y}(u) = -3\alpha_3 - 2a_1 \sin(u - u_0) - 2b_1 \cos(u - u_0) - v_y^{(1)} \tag{62}$$

$$\dot{z}(u) = a_2 \cos(u - u_0) - b_2 \sin(u - u_0) - v_z^{(1)} \tag{63}$$

and the relationship between the elements and Cartesian initial conditions are likewise modified.

This formalism results in a rather unique form for the perturbing Hamiltonian in Eq. (60); that is, $\mathcal{H}^{(1)}$ depends upon the canonical momenta and velocities. It is shown in [14,15] that for problems where the perturbing Hamiltonian is velocity dependent, the resulting instantaneous trajectory (Eqs. (28)-(30)) is not osculating. While the physical trajectory we find will be exact (and quite useful), we must therefore keep in mind that it is not formed from a series of tangent ellipses as in the other perturbation cases we are examining or as in the Delaunay formulation of the two-body problem. An alternative approach is to resolve the H-J equation for the new canonical momenta; this is a formidable task, however, and we defer to future work.

As before, we form the perturbing Hamiltonian and use Hamilton's equations to find the variational equations, which, to first-order in J_2 and the contact elements, are,

$$\dot{a}_1 = -\frac{3}{32} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} \cos(2i - u - u_0) - 7\cos(3u - u_0 + 2i) \\ +14\cos(3u - u_0) + 6\cos(u - u_0 - 2i) \\ +4\cos(u - u_0) - 7\cos(3u - u_0 - 2i) \\ +6\cos(u - u_0 + 2i) - 2\cos(u + u_0) \\ +\cos(u + u_0 + 2i) \end{pmatrix} \tag{64}$$

$$\dot{b}_1 = \frac{3}{32} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} -\sin(u + u_0 + 2i) - \sin(u + u_0 - 2i) \\ +14\sin(3u - u_0) - 7\sin(3u - u_0 + 2i) \\ -7\sin(3u - u_0 - 2i) \\ +6\sin(u - u_0 - 2i) + 6\sin(u - u_0 + 2i) \\ +2\sin(u + u_0) + 4\sin(u - u_0) \end{pmatrix} \tag{65}$$

$$\dot{a}_2 = \frac{3}{8} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 (\cos(2u - u_0 + 2i) - \cos(2u - u_0 - 2i)) \tag{66}$$

$$\dot{b}_2 = -\frac{3}{8} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 (\sin(2u - u_0 + 2i) - \sin(2u - u_0 - 2i)) \tag{67}$$

$$\dot{a}_3 = -\frac{3}{8} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 (\sin(2u - 2i) + \sin(2u + 2i) - 2\sin(2u)) \tag{68}$$

$$\dot{q}_3 = -3a_3 - \frac{9}{16} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} 4\cos(2u) - 2\cos(2u - 2i) \\ -2\cos(2u + 2i) + 3\cos(2i) + 1 \end{pmatrix} \tag{69}$$

As in the equatorial case, these equations can easily be solved by quadrature. We again find a secular drift term proportional to $a_3(0)$. However, because of our careful selection of the drifting reference orbit, we are able to find a straightforward boundedness condition,

$$a_3(u_0)_{J_2} = \frac{3}{16} J_2 \left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{bmatrix} 1 + 3\cos(2i) + 2\cos(2u_0) \\ -\cos(2i - 2u_0) - \cos(2i + 2u_0) \end{bmatrix} \tag{70}$$

Substituting this condition into the solution for the elements and then into the Cartesian generating equations result in the periodic equations for the relative motion trajectory,

$$x(u) = a_1(u_0)\sin(u-u_0) + b_1(u_0)\cos(u-u_0)$$
$$+\frac{1}{32}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} 4\cos(2u) - 2\cos(2u+2i) - 2\cos(2u-2i) \\ +12\cos(u-u_0) + 6\cos(u+u_0) \\ +18\cos(u-u_0-2i) + 18\cos(u-u_0+2i) \\ -3\cos(u+u_0-2i) - 3\cos(u+u_0+2i) \\ +14\cos(u-3u_0) \\ -7\cos(u-3u_0+2i) - 7\cos(u-3u_0-3i) \end{pmatrix} \quad (71)$$
$$y(u) = q_3(u_0) + 2a_1(u_0)\cos(u-u_0) - 2b_1(u_0)\sin(u-u_0)$$
$$+\frac{1}{32}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} 2\sin(2u) - \sin(2u-2i) - \sin(2u+2i) \\ -24\sin(u-u_0) - 12\sin(u+u_0) - 18\sin(2u_0) \\ +9\sin(2u_0+2i) + 9\sin(2u_0-2i) \\ -36\sin(u-u_0-2i) - 36\sin(u-u_0+2i) \\ +6\sin(u+u_0-2i) + 6\sin(u+u_0+2i) \\ -28\sin(u-3u_0) \\ +14\sin(u-3u_0+2i) + 14\sin(u-3u_0-2i) \end{pmatrix} \quad (72)$$
$$z(u) = a_2(u_0)\sin(u-u_0) + b_2(u_0)\cos(u-u_0)$$
$$+\frac{3}{16}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} \cos(u+2i) - \cos(u-2i) \\ +\cos(u-2u_0+2i) - \cos(u-2u_0-2i) \end{pmatrix} \quad (73)$$

Once again, these can be verified by examining the constant offset at $i = 0$ and we again find the correct result.

In [6] we present simulations of periodic relative motion trajectories for three different inclinations. For brevity, we present only one of those here. For these simulations, we selected initial conditions on the contact epicyclic elements (with the boundedness condition on $a_3(0)$ from Eq. (70)) and then found the cartesian initial conditions in the rotating and drifting frame from,

$$x(u_0) = 2a_3(u_0) + b1(u_0) \quad (74)$$
$$y(u_0) = q_3(u_0) + 2a_1(u_0) \quad (75)$$
$$z(u_0) = b_2(u_0) \quad (76)$$

and the initial rates from Eqs. (61) - (63),

$$\dot{x}(u_0) = a1_(u_0) - v_x^{(1)}(u_0) \quad (77)$$
$$\dot{y}(u_0) = -3a_3(u_0) - 2b_1(u_0) - v_y^{(1)}(u_0) \quad (78)$$
$$\dot{z}(u_0) = a_2(u_0) - v_z^{(1)}(u_0) \quad (79)$$

These initial conditions were then rotated and translated into an inertial frame for a full nonlinear simulation. The circular reference orbit selected had an altitude of 750 km. All the initial conditions on the contact elements were set to zero except for $a_3(0)$.

Figure 2 shows an example of a bounded relative motion over 5 orbits including J_2 effects relative to a sun-synchronous reference orbit. Also in this figure is the difference between the full nonlinear simulation and the relative motion from Eqs. (71)-(73). [1] The boundedness condition works quite well, resulting in an average drift of roughly 20 m/orbit.

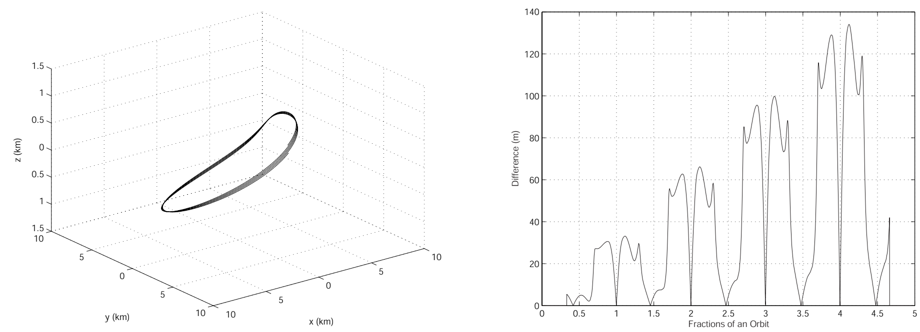

Fig. 2. *Left* Nonlinear simulation of bounded relative motion trajectory in a sun-synchronous reference orbit. *Right* A comparison of the relative displacement between the linearized trajectory and that from a full, inertial, nonlinear simulation over 5 orbits for a Sun-Synchronous reference orbit.

4 The Second-Order Oblateness Perturbation

If we keep the terms of order J_2 in the perturbing Hamiltonian, $H^{(1)}$, the error in the calculation is, as we saw in the previous section, in the order of 20-30 meters per orbit, which does not satisfy the accuracy needed for most mission analysis. To reduce the error, the terms of order J_2^2 in the zonal harmonics have to be included in the perturbation analysis. Since $J_3 \approx J_4 \approx O({J_2}^2)$, this means including the effect of J_2^2, J_3 and J_4. We consider the more complicated problem associated with the long-term period induced by J_3 in the next section, and first look at the effect of ${J_2}^2$ only.

We proceed as we did with the first-order perturbation, but now adopt a rotating frame which has the second-order effects of oblateness included,

$$\dot{\Omega} = \dot{\Omega}_{J_2} + \frac{3n{J_2}^2{R_\oplus}^4}{256\bar{r}^4}(3 + 5\cos 2i)(65 + 79\cos 2i) \tag{80}$$

$$\dot{u} = \dot{u}_{J_2} + \frac{3n{J_2}^2{R_\oplus}^4}{256\bar{r}^4}(241 - 328\cos i + 196\cos 2i - 152\cos 3i + 139\cos 4i) \tag{81}$$

[1] This figure shows the geometric difference between the two orbits. Plots of the difference in the Cartesian components show a large, and growing, oscillation due a slight difference in the rates and thus a growing offset in phasing. See [6] for details.

with a mean radius of

$$\bar{r} = \bar{a} + \frac{3J_2{R_\oplus}^2}{4\bar{a}}(3\sin^2 i - 2) - \frac{{J_2}^2{R_\oplus}^4}{32\bar{a}^3}\left(16 + 24\sin^2 i - 49\sin^4 i\right) \tag{82}$$

and in the perturbing Hamiltonian we keep the terms of order ${J_2}^2$,

$$H^{(1)} = H^{(1)}_{J_2} + H^{(1)}_{{J_2}^2}$$

where $H^{(1)}_{{J_2}^2}$ is the perturbation due to second-order oblateness perturbation.

$H^{(1)}_{{J_2}^2}$ and some of the following formulas and equations are too lengthy to be placed in this paper. Only the procedure with which these equations are obtained will be explained; for exact solutions, please contact the authors.

Recall that for the first-order solution, the condition for a bounded relative motion is given in Eqn. (70). Here, where we include the effect of the higher-order terms, we can find a new condition for boundedness that is second-and third-order in the initial conditions. We do this by solving the variational equations above using a Poincarè-Lindstedt procedure. We add to the epicyclic elements small, time varying perturbations that solve the variational equations, which are second order versions of Eqs. (64) -(69). These small, time varying terms are going to be second-order and third-order in the initial conditions. The second-order terms are found by plugging in the first-order,${a_3}_{J_2}$) into the second order differential equations and integrating by quadrature. Doing so we find that all of the epicyclic elements are periodic except for q_3. However, by setting the non-periodic part of q_3 equal to zero we find a new condition on $a_3(u_0)$ to ensure bounded relative motion,

$$a_3(u_0) = a_3(u_0)_{J_2} + \frac{J_2^2{R_\oplus}^4}{1024\bar{r}^4}\begin{pmatrix} -660\cos(4u_0)\sin^4(i) \\ +192(8\cos(2i)+11)\cos(2u_0)\sin^2(i) \\ -1684\cos(2i) - 203\cos(4i) - 1569 \end{pmatrix} \tag{83}$$

Our boundedness condition now consists of second-order powers of the initial conditions. The resulting solution for $(a_1(u), b_1(u), a_2(u), b_2(u), a_3(u), q_3(u))$ is substituted back into the Cartesian Eqs. (28) - (30) to find the complete second-order solution.

The periodic orbits around the J_2, ${J_2}^2$ mean rotating frame can be classified into three different categories.

(1) Orbits mostly in z direction with a characteristic 8 shape. These orbits correspond to the invariance of the Hamiltonian under Ω, argument of the ascending node.

(2) Earth oblateness leads to secular drift in Ω and u. The second type of orbits corresponds to the solution of the drift matching conditions, also illuminated in [10], given as:

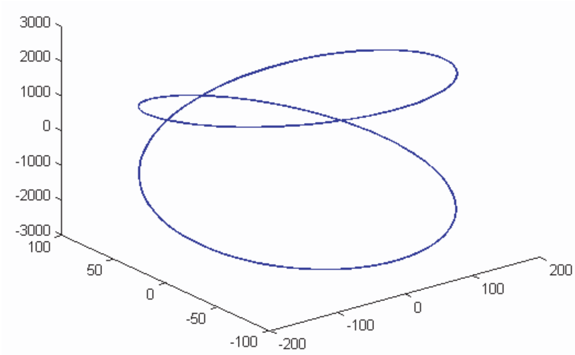

Fig. 3. Bounded Relative Motion corresponding to Ω invariance

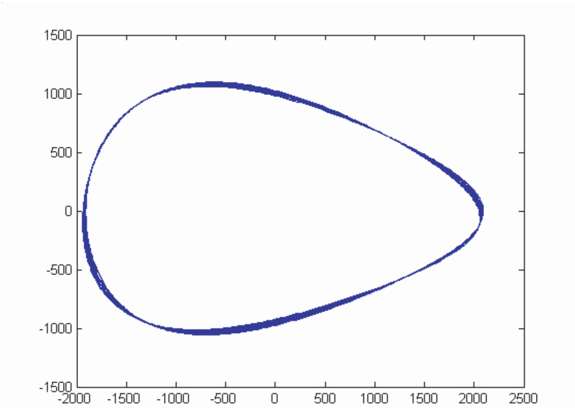

Fig. 4. Bounded Relative Motion corresponding to drift rate matching

$$\dot{\Omega}_1 = \dot{\Omega}_2 \quad \& \quad \dot{u}_1 = \dot{u}_2$$

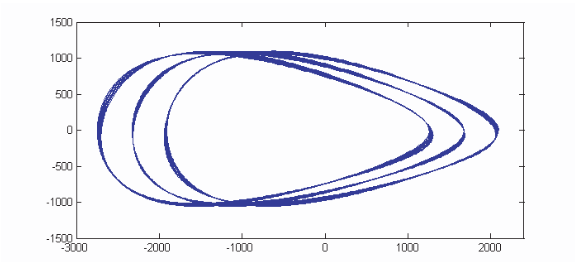

Fig. 5. Bounded Relative Motion corresponding to u invariance

(3) Orbits that are exact copies of any given periodic orbit but shifted in the y direction. These orbits correspond to the invariance under u, argument of the longitude.

An interesting connection exists between these periodic orbit families and the ones around libration points of the Restricted Three Body Problem (RTBP).

In both cases, there exists an in-plane family which is termed Horizontal Lyapunov Orbits, and an out-of-plane 8-shaped family named Vertical Lyapunov

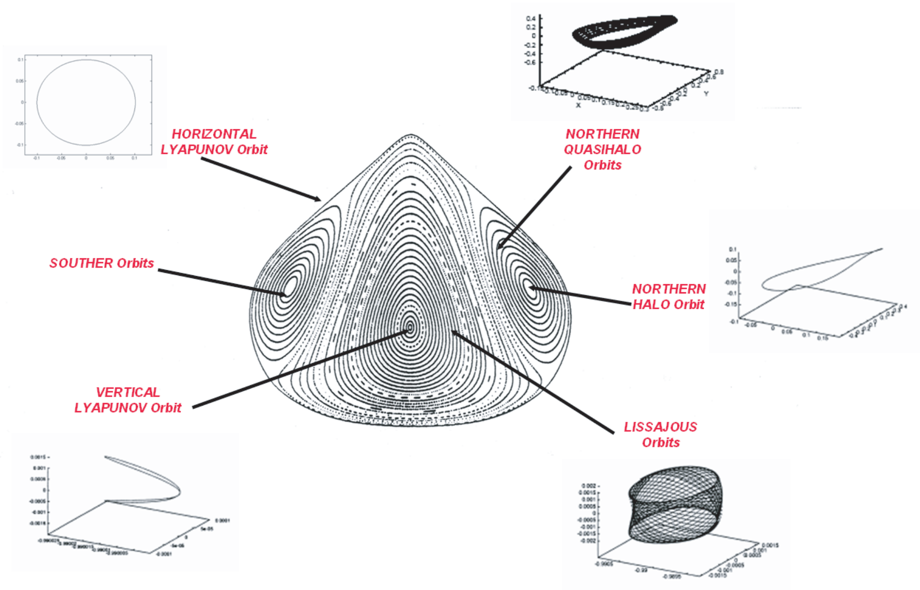

Fig. 6. Periodic Orbit Families around L_2 shown on the reduced phase space

Orbits with Lissajous Quasiperiodic Orbits around them. This resemblance is due to the effect of the centrifugal force that appears when we utilize a rotating frame, and to the comparable effects of extra pull induced by J_2 and third-body gravitational potentials. This shows that these techniques may in the future be fruitfully applied to libration point orbit analysis.

Further analysis of the phase space for the oblateness perturbation problem, which we leave for future work, may result in Halo-like periodic orbits.

Two approaches exist for our aim of placing satellites in periodic orbits around each other. The first approach is to choose a mean rotating orbit and place the satellites in any of the three families of periodic orbits defined before by using the boundedness condition. Using this approach, the closed-form solution of the relative motion is obtained by mere subtraction of the position of each satellite relative to the rotating frame. The second approach is to move the reference frame from the mean fictitious orbit to a real orbit by adding the second-order linear solution and then solving the boundedness condition around this orbit. The two procedures are shown in Figure 7.

Figure 8 shows a nonlinear simulation of a 3-dimensional relative motion trajectory between two satellite orbits with an altitude of 500 km and an inclination of 20 degrees for over 20 orbits. The second-order boundedness condition was used, resulting in an overall drift of 50 cm/orbit. On the right, figure 8 shows the difference between the nonlinear simulation and the second-order variational solution.

One notices the so-called "tumbling", [21], of the relative orbit due to the short-term variation in the argument of periapsis. Observed in many papers, this effect leads to a distortion of the x-z and y-z projections of the motion from

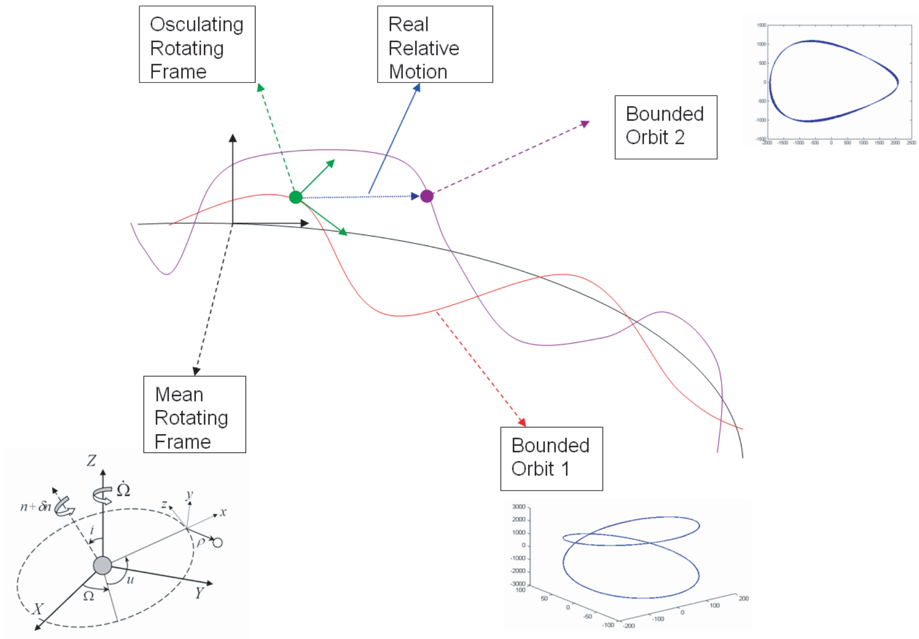

Fig. 7. Reference frames used to define relative motion

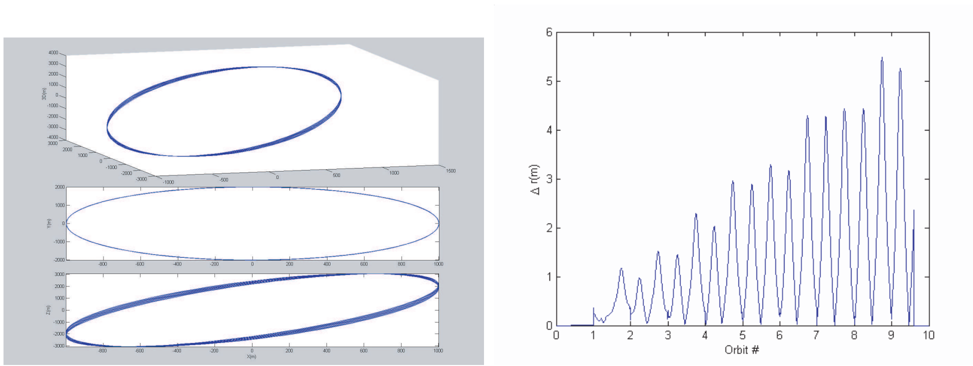

Fig. 8. *Left* Nonlinear simulation of bounded relative motion trajectory between two satellites for 20 orbits in a 30^o inclination reference orbit. *Right* Relative motion comparison between the linearized equations and a full, inertial, nonlinear simulation over 10 orbits for a 20^o inclination reference orbit.

an ellipse to a degenerate ellipse and back to an ellipse again. The tumbling rate of the ellipse is equal to the rate of change of the argument of periapsis. Three examples from other papers with results comparable to our own are shown in Figure 9.

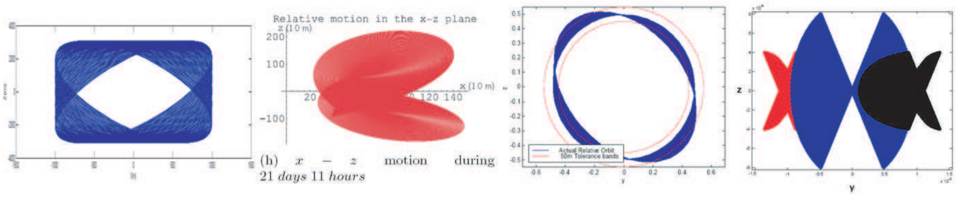

Fig. 9. Tumbling of the relative orbit examples . From left to right, our result, Guibout et al. [22], Schaub et al. [3], Koon et al. [23]

5 J_2, J_3 and J_4 Zonal Perturbation

As the rotating frame, we here use the well-known SGP4 analytic orbit propagator's, [19], long-term solution, which includes the secular effects of J_2, ${J_2}^2$ and J_4 and the long-term periodic effect of J_3. When we take into account the perturbing effects of J_3 and J_4 and apply the same formalism as in Section 4, the boundedness condition increases from 1 to 5. There thus exists only one osculating orbit that stays bounded around the mean rotating frame. The only freedom left is a δu offset. This severe constraint of motion is due to the long-term variation in J_3.

By relaxing some of these constraints and solving Hamilton's equations using a Poincarè-Lindstedt procedure, we obtain approximations for the almost periodic orbits.

Figure 10 shows a nonlinear simulation of relative motion trajectories between two satellite orbits, with an altitude of 500 km and an inclination of 20 degrees, for over 20 orbits. In this simulation, 4 instead of 5 boundedness conditions were used, resulting in an overall drift of 30 cm/orbit. Figure 10 shows the difference between the nonlinear simulation and the second-order variational solution.

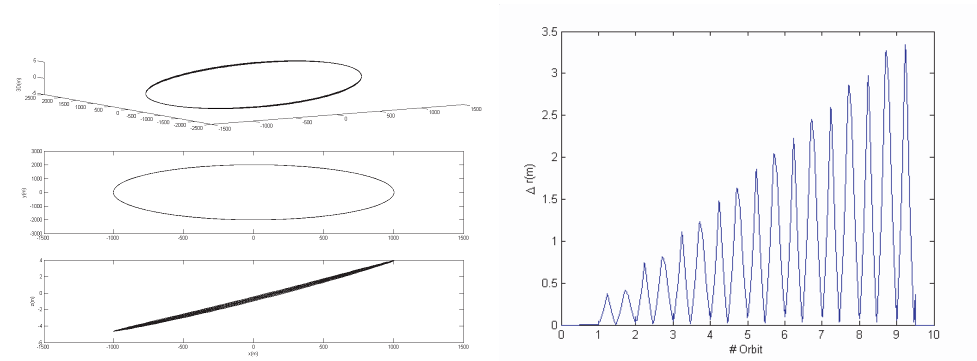

Fig. 10. *Left* Nonlinear simulation of bounded relative motion trajectory between two satellites for 20 orbits in a 30^o inclination reference orbit. *Right* Relative motion comparison between the linearized equations and a full, inertial, nonlinear simulation over 10 orbits for a 20^o inclination reference orbit.

6 Conclusions

We summarized a new framework for modelling relative motion about circular reference orbits. We reformulated the well-known 2:1 elliptical solution of the CW equations into a form dependent upon six canonical constants of the motion that are easily related to the Cartesian initial conditions in the rotating frame. We then use canonical perturbation theory to find variational equations for these elements, which we termed epicyclic, in direct analogy to the variation of the orbital elements. Not only does this approach provide straightforward, first-order differential equations of variation, but the description of the motion remains entirely in the relative motion frame, where most measurements are taken and where trajectory specification is most natural. In this paper, we demonstrated this technique by finding conditions for bounded, periodic motion in the presence of zonal induced perturbations. We were also able to find a general expression for J_2, J_3 and J_4 invariant orbits at any inclination and altitude. There is much that can still be done to extend this methodology. In particular, we are interested in pursuing the combined effects of eccentricity and high-order zonal harmonics. We also intend to explore control techniques incorporating a control potential. Finally, we believe this approach can be fruitfully applied to motion around the co-linear Lagrange points in the circular restricted three-body problem.

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